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Dynamic agglomeration patterns
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Abstract

We introduce a 2-country New Economic Geography model with 4 regions. It is defined by a 2D piecewise smooth map that depends on 8 parameters. Using reductions of this map to 1D maps defined on invariant straight lines we obtain stability conditions of the Core-Periphery fixed points, and show how such reductions can be used to describe basins of attraction of coexisting attractors. Typical bifurcation sequences obtained when varying some parameters are discussed. We find patterns that are much richer than those observed in standard NEG models: there are more types of fixed points including fixed points attracting in Milnor’s sense; their basins of attraction are quite complicated; and coexistence is pervasive.

1 Introduction

Models of the so-called New Economic Geography (NEG) - pioneered by Krugman in [9] - are now widely used to explain patterns of economic activity across space. In particular, these models aim to explain how economic activity may end up agglomerating in only a few regions although all regions

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are assumed to be identical ex-ante. In these models, commodities are traded between regions at some trade costs and some productive factors are allowed to migrate across regions. Indeed, factor mobility governs the dynamic process that determines the spatial distribution of economic activity. This process involves agglomerating and spreading forces. The main agglomerating force is based on the access to larger markets: Due to the monopolistically competitive market structure, larger markets offer higher factor rewards and attract factor migration. The price index effect works in the same direction: as more product varieties are locally available in larger markets, migration is favoured towards these locations. The main spreading force is the competition effect: The larger a market, the more intense the local competition given the higher number of firms in this market. Another dispersion force comes from the regional immobility of part of the labor force (e.g. that employed in the agricultural sector), a typical assumption made in the NEG literature that introduces a minimum size below which the local market cannot shrink.

In standard NEG models, the agglomerating and spreading forces get stronger when trade costs fall, with agglomerating forces dominating spreading ones for low trade costs. In addition, a typical NEG story puts much weight on the existence of a range of trade costs in which the dispersed and the agglomerated equilibrium coexist and in which historic contingencies determine the long-run regional distribution of the economic activity.

Despite the popularity of NEG models, several drawbacks limit their relevance.

First, most NEG models comprise just two regions or countries, where each country is taken as a whole. Only recently, models with more regions have been proposed in the literature. For example, in [10], [12], [3] and [7] models are put forward with three regions, where two regions belong to the same country and the third one represents a separate country; closer to the set-up used in this paper, [14] and [2] consider models with two countries where each of them is split symmetrically into two regions. In this paper, similarly to these latter contributions, we also study a model that comprises two countries, each of which consisting of two regions. For a comprehensive review on multiregional NEG models, the reader can refer to [8].

Second, most NEG models depict a very simple geographical structure: regions are separated by symmetric trade costs; this is also the case for 3-region NEG models, see e.g., [1] and [4] for core-periphery models with three regions evenly distributed along a line or a circle respectively. Our model introduces some non-homogeneity in the geographical structure: the four regions are arranged on a
line and the two countries share a common border involving only the two central regions. Moreover, due to various reasons (i.e. topographical characteristics, limited information, and so on), trade takes place only between adjacent regions. It follows that the two peripheral (interior) regions do not have access to the foreign market and have only one trading partner which belongs to the same country, whereas the two central (border) regions engage in both interregional and international trade.

Third, while most NEG models are set in continuous time, we prefer a discrete time approach that is better suited to depict delays in factor migration processes. As a result, the dynamics generated by our model is much richer.

In this paper, we explicitly analyse the dynamic processes governing a two-country NEG model with four regions by providing analytical results and further insights gained from simulations. We detect a sequence which corresponds to that found in standard two-country NEG models: dispersion for high trade costs and agglomeration for low trade costs. However, the patterns we find are much richer. First, there is a much larger variety of fixed point attractors, dispersion is no longer symmetric, and agglomeration can take several forms, including asymmetric ones. More interestingly and going beyond standard NEG results, partial agglomeration is also a possible outcome. Second, we find that coexistence of equilibria is much more pervasive than standard NEG models suggest. Third, our analysis reveals that some fixed points are attracting in Milnor’s sense, involving complex basins of attraction and a high sensitivity to initial conditions.

The paper is organised as follows. In Sec.2 we describe basic economic ingredients and assumptions leading to the considered NEG model. The model is defined by a two-dimensional (2D) piecewise smooth map $Z$ depending on 8 parameters. Peculiarity of the map $Z$ is related to the presence of flat branches in its definition, which are responsible for the appearance of Milnor attractors. Local and global dynamics of the map $Z$ in the symmetric case are studied in Sec.3. In particular, fixed points of $Z$ are analysed in Sec.3.1. Given that the analytical expression of the map $Z$ is quite complicated we use reductions of $Z$ to 1D piecewise smooth maps defined on invariant straight lines. These reductions help to obtain stability conditions of the Core-Periphery fixed points as well as to describe some global dynamic properties of $Z$, for example, basins of attraction of coexisting attractors (see Sec.3.2). In Sec.3.3 we discuss typical bifurcation scenarios observed in the model under variation of different parameters. Sec.4 concludes.
2 Model

2.1 The economy

The economy consists of two countries (\( H = \text{Home} \) and \( F = \text{Foreign} \)), each of which having 2 regions: regions 1 and 2 compose the \( H \)-country and regions 3 and 4 the \( F \)-country. Natural geography is assumed to distribute regions along a line from 1 to 4: so that the “border” regions 2 and 3 – in the middle of the line – are on the two sides of the international border and each of them is connected with an inland or “interior” region, 1 and 4, respectively (see Fig.1). We assume that the interior regions 1 and 4 do not have any access to the international market meaning that international trade is limited to exchanges between the border regions 2 and 3.\(^1\) The freeness of internal trade in the Home country (resp. the Foreign country) is denoted by \( \phi_{Ih} \) (resp. \( \phi_{If} \)); this parameter is inversely related to domestic transport costs; \( \phi_{Ih} = 0 \) (resp. \( \phi_{If} = 0 \)) denotes prohibitively high trade costs and \( \phi_{Ih} = 1 \) (resp. \( \phi_{If} = 1 \)) indicates internal free trade; the freeness of external trade – inversely related to international trade barriers – is analogously defined and denoted by \( \phi_E \).

There are two sectors in the economy: manufacturing, which exhibits increasing returns to scale and agriculture, which has constant returns. There are two types of workers. Unskilled labor is geographically immobile whereas skilled workers (i.e. entrepreneurs) can relocate from one region to another within their country. No migration occurs between countries. Each region is endowed with \( L/4 \) unskilled labor and both countries have the same number of mobile entrepreneurs \( E/2 \).

The utility function of a representative consumer, unskilled worker or entrepreneur, is given by

\[
U = C_M^{\mu}C_A^{1-\mu},
\]

where \( C_A \) is the consumption of the agricultural good and \( C_M \) the consumption of a CES composite of manufactured varieties given by

\[
C_M = \left[ \sum_{i=1}^{n} c_i^{(\sigma-1)/\sigma} \right]^{\sigma/(\sigma-1)}
\]

\(^1\)This could translate the fact that the availability of information is geographically limited. In this case, trade only occurs when direct knowledge of the local market is available (manufactures in a border region get this information from the other border region but not from the interior region in the other country). Alternatively, geographical barriers could be such that inland transport of foreign commodities involves particularly high costs, a situation that we approximate by no-trade.
with $c_i$ representing the consumption of variety $i$, $n$ the number of available varieties, $\sigma > 1$ the elasticity of substitution across varieties and $0 < \mu < 1$ the share of manufacturing expenditure.

The budget constraint faced by a consumer is

$$\sum_{i=1}^{n} p_i c_i + p_A C_A = y,$$

where $p_A$ is the price of the homogeneous agricultural good, $\tilde{p}_i$ the price of variety $i$ inclusive of trade costs and $y$ the individual’s income.

The agricultural sector is perfectly competitive and production requires 1 unit of unskilled labor for the production of 1 unit of output.

The manufacturing sector is monopolistically competitive: identical firms produce differentiated varieties using a technology requiring a fixed component (an entrepreneur) and a variable component (unskilled labor) with $\beta$ units of labor for each additional unit of output. The total cost of producing $q_i$ units of variety $i$ is given by

$$TC(q_i) = \pi_i + w\beta q_i,$$

where the entrepreneur remuneration $\pi_i$ represents the fixed cost component.

The short-run equilibrium in period $t$ is defined in terms of the spatial distribution of entrepreneurs across regions in that period. The share of entrepreneurs of the Home country (resp. the Foreign country) in region 2 (resp. region 3) is denoted by $x_t$ (resp. $y_t$).

Because of zero agricultural profits, $w = p_A = 1$ by choosing the agricultural good as numeraire. Each manufacturing firm faces a demand curve with a constant elasticity $\sigma$ and optimally sets the
price of its variety at a fixed markup over marginal cost

\[ p = \frac{\sigma}{\sigma - 1} \beta. \]

The income in each region is given by

\[ Y_{1,t} = L/4 + \pi_{1,t}(1 - x_t)E/2; \quad Y_{2,t} = L/4 + \pi_{2,t}x_tE/2; \]
\[ Y_{3,t} = L/4 + \pi_{3,t}y_tE/2; \quad Y_{4,t} = L/4 + \pi_{4,t}(1 - y_t)E/2. \]  \(1\)

The demand facing a firm located in each region can be written as

\[ d_{1,t} = \mu \left( Y_{1,t}P_{1,t}^{\sigma - 1} + \phi_{1h}Y_{2,t}P_{2,t}^{\sigma - 1} \right) /p^\sigma; \quad d_{2,t} = \mu \left( \phi_{1h}Y_{1,t}P_{1,t}^{\sigma - 1} + Y_{2,t}P_{2,t}^{\sigma - 1} + \phi_EY_{3,t}P_{3,t}^{\sigma - 1} \right) /p^\sigma; \]
\[ d_{3,t} = \mu \left( \phi_EQ_{2,t}P_{2,t}^{\sigma - 1} + Y_{3,t}P_{3,t}^{\sigma - 1} + \phi_{1f}Y_{4,t}P_{4,t}^{\sigma - 1} \right) /p^\sigma; \quad d_{4,t} = \mu \left( \phi_{1f}Y_{3,t}P_{3,t}^{\sigma - 1} + Y_{4,t}P_{4,t}^{\sigma - 1} \right) /p^\sigma; \]  \(2\)

where \( P_{r,t} \) denotes the manufacturing price index in region \( r \):

\[ P_{r,t} = p(E/2)^{1/(1-\sigma)} \Delta_{r,t}^{1/(1-\sigma)} \]

with \( \Delta_{r,t} \) given by

\[ \Delta_{1,t} = 1 - (1 - \phi_{1h})x_t; \quad \Delta_{2,t} = \phi_{1h} + (1 - \phi_{1h})x_t + \phi_Ey_t; \]
\[ \Delta_{3,t} = \phi_{1f} + (1 - \phi_{1f})y_t + \phi_EQ_{t}; \quad \Delta_{4,t} = 1 - (1 - \phi_{1f})y_t. \]

The entrepreneur remuneration in region \( r \) is given by

\[ \pi_{r,t} = pq_{r,t} - \beta q_{r,t} = pq_{r,t}/\sigma, \]

where \( q_{r,t} \) is the supply of a firm in region \( r \).

Since in equilibrium, \( d_{r,t} = q_{r,t} \), the entrepreneur remuneration in each region can be rewritten by using expression (2)

\[ \pi_{1,t} = \frac{2\mu}{\sigma E} \left( \frac{Y_{1,t}}{\Delta_{1,t}} + \frac{\phi_{1h}}{\Delta_{2,t}} \right); \quad \pi_{2,t} = \frac{2\mu}{\sigma E} \left( \frac{\phi_{1h}Y_{1,t}}{\Delta_{1,t}} + \frac{Y_{2,t}}{\Delta_{2,t}} + \frac{\phi_EQ_{t}}{\Delta_{3,t}} \right); \]
\[ \pi_{3,t} = \frac{2\mu}{\sigma E} \left( \frac{\phi_EQ_{2,t}}{\Delta_{2,t}} + \frac{Y_{3,t}}{\Delta_{3,t}} + \frac{\phi_{1f}Y_{4,t}}{\Delta_{4,t}} \right); \quad \pi_{4,t} = \frac{2\mu}{\sigma E} \left( \frac{\phi_{1f}Y_{3,t}}{\Delta_{3,t}} + \frac{Y_{4,t}}{\Delta_{4,t}} \right). \]

By plugging the income expression (1) into the above entrepreneur remuneration, and then solving for \( \pi_{r,t} \) we get

\[ \pi_{1,t} = \frac{L}{2E} \left( \frac{1}{\Delta_{1,t}} + \frac{\phi_{1h}}{\Delta_{2,t}} \right) + \phi_{1h}x_t \frac{\pi_{2,t}}{\Delta_{2,t}}; \quad \pi_{2,t} = \frac{A_{f,t}B_{h,t} + B_{f,t}C_{h,t}}{A_hA_f - C_hC_{f,t}}. \]
\[ \pi_{3,t} = A_{h,t} B_{f,t} + B_{h,t} C_{f,t}; \quad \pi_{4,t} = \frac{L}{2E} \left( \frac{1}{\Delta_{2,t}} + \frac{\phi_{ft} y_t}{\Delta_{3,t}} \right) + \frac{\phi_{ft} y_t}{\Delta_{3,t}} \]

where

\[
A_{h,t} = \frac{\sigma}{\mu} - \frac{x_t}{\Delta_{2,t}} \left( 1 + \frac{\phi_{ft}^2 (1 - x_t)}{\mu \Delta_{1,t} - 1 + x_t} \right); \quad B_{h,t} = \frac{L}{2E} \left[ \frac{\sigma \Delta_{2,t} + \phi_{ft} (1 - x_t)}{\Delta_{2,t} (\frac{\sigma}{\mu} \Delta_{1,t} - 1 + x_t)} \phi_{ft} + \frac{\phi_{E}}{\Delta_{2,t}} + \frac{1}{\Delta_{2,t}} \right];
\]

\[
A_{f,t} = \frac{\sigma}{\mu} - \frac{y_t}{\Delta_{3,t}} \left( 1 + \frac{\phi_{ft}^2 (1 - y_t)}{\sigma \Delta_{4,t} - 1 + y_t} \right); \quad B_{f,t} = \frac{L}{2E} \left[ \frac{\sigma \Delta_{3,t} + \phi_{ft} (1 - y_t)}{\Delta_{3,t} (\frac{\sigma}{\mu} \Delta_{4,t} - 1 + y_t)} \phi_{ft} + \frac{\phi_{E}}{\Delta_{2,t}} + \frac{1}{\Delta_{2,t}} \right];
\]

\[ C_{h,t} = \frac{\phi_{E} y_t}{\Delta_{3,t}}; \quad C_{f,t} = \frac{\phi_{E} x_t}{\Delta_{2,t}}. \]

From the above expressions, the indirect utilities \( V_{r,t} = \pi_{r,t} / P_{r,t} \) can also be formulated in terms of \( x_t \) and \( y_t \).

### 2.2 Dynamic equations

Entrepreneurs have an incentive to relocate to the region in their country providing them with the highest indirect utility. The migration process resembles the evolutionary replicator dynamics leading to a family of 2D piecewise smooth maps \( Z \) which can be written, skipping the time index \( t \), as follows:

\[
Z : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} 0 & \text{if } Z_h(x, y) < 0, \\ Z_h(x, y) & \text{if } 0 \leq Z_h(x, y) \leq 1, \\ 1 & \text{if } Z_h(x, y) > 1, \\ 0 & \text{if } Z_f(x, y) < 0, \\ Z_f(x, y) & \text{if } 0 \leq Z_f(x, y) \leq 1, \\ 1 & \text{if } Z_f(x, y) > 1, \end{cases}
\]

where

\[
Z_h(x, y) = x \left[ 1 + \gamma (1 - x) \right] \frac{V_2 - V_1}{(1 - x)V_1 + xV_2} = x \left[ 1 + \gamma (1 - x) \right] \frac{\Omega_h(x, y) - 1}{1 + x(\Omega_h(x, y) - 1)};
\]

\[
Z_f(x, y) = y \left[ 1 + \gamma (1 - y) \right] \frac{V_3 - V_4}{(1 - y)V_4 + yV_3} = y \left[ 1 + \gamma (1 - y) \right] \frac{\Omega_f(x, y) - 1}{1 + x(\Omega_f(x, y) - 1)};
\]

\[
\Omega_h(x, y) = \frac{V_2}{V_1}; \quad \Omega_f(x, y) = \frac{V_3}{V_4}; \quad (4)
\]

with \( \gamma > 0 \) representing the migration speed.
The map $Z$ depends on 8 parameters, namely, $\gamma, \mu, \sigma, \phi_{ih}, \phi_{if}, \phi_E, E, L$. In the present paper we focus on the symmetric case

$$\phi_{ih} = \phi_{if} := \phi.$$ 

The expressions in (4) in such a case can be rewritten in the following form:

$$\Omega_h(x, y) = \frac{\pi_2}{\pi_1} \left( \frac{\Delta_2}{\Delta_1} \right)^{\mu / \sigma - 1}, \quad \Omega_f(x, y) = \frac{\pi_3}{\pi_4} \left( \frac{\Delta_3}{\Delta_4} \right)^{\mu / \sigma - 1},$$

where

$$\Delta_1 = 1 - (1 - \phi)x, \quad \Delta_2 = \phi + (1 - \phi)x + \phi Ey,$$

$$\Delta_3 = \phi + (1 - \phi)y + \phi Ex, \quad \Delta_4 = 1 - (1 - \phi)y,$$

$$\pi_1 = \frac{L}{2E} \left( \frac{1}{\Delta_1} \right) + \phi \pi_2, \quad \pi_4 = \frac{L}{2E} \left( \frac{1}{\Delta_4} \right) + \phi \pi_3, \quad \pi_2 = \frac{A_fB_h + B_fC_h}{A_hA_f - C_hC_f}, \quad \pi_3 = \frac{A_hB_f + B_hC_f}{A_hA_f - C_hC_f},$$

$$A_h = \frac{\sigma}{\mu} - \frac{x}{\Delta_2} \left( 1 + \frac{\phi^2(1 - x)}{\Delta_1 - 1 + x} \right), \quad B_h = \frac{L}{2E} \left( \frac{\sigma}{\mu} \Delta_2 + \phi(1 - x) \frac{\Delta_1 - 1 + x}{\Delta_2} \frac{\phi + \phi_E + 1}{\Delta_2} \right),$$

$$A_f = \frac{\sigma}{\mu} - \frac{y}{\Delta_3} \left( 1 + \frac{\phi^2(1 - y)}{\Delta_4 - 1 + y} \right), \quad B_f = \frac{L}{2E} \left( \frac{\sigma}{\mu} \Delta_3 + \phi(1 - y) \frac{\Delta_4 - 1 + y}{\Delta_3} \frac{\phi + \phi_E + 1}{\Delta_3} \right),$$

$$C_h = \frac{y\phi_E}{\Delta_3}, \quad C_f = \frac{x\phi_E}{\Delta_2}.$$ 

3 Local and global dynamics

Our aim is to study the dynamic properties of the map $Z$ and how these properties are affected by parameters of interest. It is clear that due to the constraints in the definition of $Z$ we consider only the positive unit square of the phase plane, that is, the set $I^2 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \ 0 \leq y \leq 1\}$.

Let the border lines of $I^2$ and the main diagonal of the phase plane be denoted as follows:

$$L_{0,y} = \{(x, y) : x = 0\}, \quad L_{x,0} = \{(x, y) : y = 0\},$$

$$L_{1,y} = \{(x, y) : x = 1\}, \quad L_{x,1} = \{(x, y) : y = 1\},$$

$$L_d = \{(x, y) : x = y\}.$$ 

Below in all the numerical simulations we fix

$$E = 100, \quad L = 400, \quad \mu = 0.5$$

and study how the dynamics depends on the parameters $\gamma, \sigma, \phi$, and $\phi_E$. 

8
3.1 Fixed points

Due to the symmetry of the map $Z$ with respect to the main diagonal $L_d$, any invariant set $A$ of $Z$ is either symmetric with respect to $L_d$ or there exits one more invariant set $A'$ symmetric to $A$. This property applies, in particular, to the fixed points of $Z$, each of which either belongs to $L_d$, or a couple of fixed points exist symmetric to each other with respect to $L_d$.

The simplest fixed points of $Z$ are *Core-Periphery fixed points* denoted as follows:

$$\begin{align*}
CP_{00} : (x, y) = (0, 0), & \quad CP_{11} : (x, y) = (1, 1), \\
CP_{01} : (x, y) = (0, 1), & \quad CP_{10} : (x, y) = (1, 0).
\end{align*}$$

As we shall see, the map $Z$ can also have *interior symmetric fixed points*,

$$S_{aa} : (x, y) = (a, a),$$

(with $a \neq 0$, $a \neq 1$), *interior asymmetric fixed points* (necessarily generated in pairs due to the symmetry of $Z$ mentioned above),

$$AS_{ab} : (x, y) = (a, b), \quad AS_{ba} : (x, y) = (b, a),$$

(with $a, b \neq 0$ and $a, b \neq 1$), and *border asymmetric fixed points*,

$$AS_{0a} : (x, y) = (0, a), \quad AS_{a0} : (x, y) = (a, 0),$$
$$AS_{1a} : (x, y) = (1, a), \quad AS_{a1} : (x, y) = (a, 1),$$

(with $a \neq 0$ and $a \neq 1$).

In fact, solving for the fixed points of the map $Z$ one gets that the interior symmetric and asymmetric fixed points satisfy the following system of equations:

$$\begin{align*}
\Omega_h(x, y) = 1, \\
\Omega_f(x, y) = 1,
\end{align*}$$

while each of the border asymmetric fixed point satisfies the related system of equations:

$$\begin{align*}
\Omega_f(x, y) = 1, \\
x = 0, \\
\Omega_h(x, y) = 1, \\
y = 0, \\
\Omega_f(x, y) = 1, \\
x = 1, \\
\Omega_h(x, y) = 1, \\
y = 1.
\end{align*}$$
These facts help us to visualise the location of fixed points, namely, any fixed point of $Z$, except the CP fixed points, is either an intersection point of the curves

\[ \Omega_h = \{(x, y) \in I^2 : \Omega_h(x, y) = 1\}, \tag{5} \]

\[ \Omega_f = \{(x, y) \in I^2 : \Omega_f(x, y) = 1\}, \tag{6} \]

or is an intersection point of the curve $\Omega_h$ with $L_{x,0}$ or $L_{x,1}$, or of the curve $\Omega_f$ with $L_{0,y}$ or $L_{1,y}$.

It is clear that for a border fixed point only one-side eigenvalues are defined (obviously, two among four one-side eigenvalues equal 0 at a CP fixed point, and one among two one-side eigenvalues equal 0 at border asymmetric fixed points). When discussing the stability of a border fixed point or some other invariant set belonging to the border of $I^2$, we mean the related one-side stability. For example, for short we call a CP fixed point an attracting/repelling node or saddle based on its non-zero one-side eigenvalues.

### 3.2 Reduction of 2D map $Z$ to 1D maps on invariant lines

It is easy to see that the border lines $L_{x,0}, L_{x,1}, L_{0,y}, L_{1,y}$, as well as the main diagonal $L_d$, are invariant under the map $Z$ which on each of these lines is reduced to a corresponding 1D map. The dynamics description of these maps can serve as a frame for the investigation of the overall dynamics of the map $Z$. Note that due to the symmetry of $Z$ it is enough to consider its reduction to the lines $L_d, L_{x,0}$ and $L_{x,1}$ only.

So, let first $(x, y) \in L_d$. The map $Z$ on $L_d$ is reduced to the following 1D piecewise smooth map:

\[ z_d : x \mapsto \begin{cases} 
0 & \text{if } z_d(x) < 0, \\
z_d(x) & \text{if } 0 \leq z_d(x) \leq 1, \\
1 & \text{if } z_d(x) > 1,
\end{cases} \]

where

\[ z_d(x) = x + \gamma x (1 - x) \frac{\Omega(x) - 1}{1 + x(\Omega(x) - 1)}, \quad \Omega(x) = \frac{\pi_2}{\pi_1} \left( \frac{\Delta_2}{\Delta_1} \right)^{\mu/(\sigma - 1)}, \]

\[ \Delta_1 = 1 - (1 - \phi)x, \quad \Delta_2 = \phi + (1 - \phi + \phi_E)x, \]

\[ \pi_1 = \frac{L}{2E} \left( \frac{1}{\Delta_1} + \phi \frac{1}{\Delta_2} \right) + \phi x \frac{\pi_2}{\Delta_2}, \quad \pi_2 = \frac{B}{A - C}, \]
\[ A = \frac{\sigma}{\mu - x} \left(1 + \frac{\phi^2(1-x)}{\frac{\alpha}{\mu} \Delta_2 - 1 + x}\right), \quad B = \frac{L}{2E} \left(\frac{\phi \Delta_2 + \phi^2(1-x)}{\Delta_2 \left(\frac{\alpha}{\mu} \Delta_2 - 1 + x\right)} + \frac{\phi E + 1}{\Delta_1}\right), \quad C = \frac{x \phi E}{\Delta_2}. \]

The map \( z_d \) obviously has the CP fixed points

\[ CP_0 : \quad x = 0, \quad CP_1 : \quad x = 1. \]

Besides the CP fixed points, \( z_d \) can also have two more fixed points,

\[ S_a : \quad x = a, \quad S_b : \quad x = b, \]

with \( a, b \neq 0, a, b \neq 1 \), which are solutions to the equation \( \Omega(x) = 1 \), that is,

\[ \frac{\frac{\alpha}{\mu} \phi \Delta_2 + \phi^2(1-x) + (\phi E + 1) \left(\frac{\alpha}{\mu} \Delta_1 - 1 + x\right)}{\frac{\alpha}{\mu} (\Delta_2 + \phi \Delta_1) - x(1 + \phi E) + \phi x} \left(\frac{\Delta_2}{\Delta_1}\right)^{\mu/(\sigma-1)} = 1. \]

Note that \( x = 0.5 \) is a fixed point of \( z_d \) only for \( \phi_E = 0 \). In fact, substituting \( x = 0.5 \) to (7) we get the equality

\[ \left(1 + \frac{2\phi \phi_E \sigma}{\phi^2(\sigma + \mu) + (\phi E + 1)(\sigma - \mu) + 2\phi \sigma}\right) \left(1 + \frac{\phi_E}{1 + \phi}\right)^{\mu/(\sigma-1)} = 1, \]

which can be satisfied only for \( \phi_E = 0 \).

Let us check the stability of the CP fixed points of \( z_d \). The right-side derivative of \( z_d \) at \( CP_0 \) is

\[ z'_d(0+) = 1 + \gamma(\Omega(0) - 1). \]

Due to the zero-constraint of the map the inequality \( z'_d(0+) \geq 0 \) always holds, while the inequality \( z'_d(0+) \leq 1 \) holds for \( \Omega(0) \leq 1 \), from which, we get stability condition for \( CP_0 \):

\[ \frac{(\sigma + \mu) \phi^2 + (\phi_E + 1)(\sigma - \mu)}{2\sigma \phi} \phi^{\mu/(\sigma-1)} < 1. \]

The equality in (8) defines a boundary in the parameter space of the map \( z_d \), denoted \( BT_{d0} \): 

\[ BT_{d0} : \quad \frac{(\sigma + \mu) \phi^2 + (\phi_E + 1)(\sigma - \mu)}{2\sigma \phi} \phi^{\mu/(\sigma-1)} = 1. \]

When crossing that boundary, a border-transcritical (BT for short) bifurcation occurs and the fixed point \( CP_0 \) loses stability.

The left-side derivative of \( z_d \) at the fixed point \( CP_1 \) is

\[ z'_d(1-) = 1 - \gamma \frac{\Omega(1) - 1}{\Omega(1)}. \]
Due to the unity-constraint of the map, the inequality \( z'_d(1_-) \geq 0 \) always holds, while \( z'_d(1_-) \leq 1 \) holds for \( \Omega(1) \geq 1 \), so that the stability condition of \( CP_1 \) is
\[
\frac{2\sigma \phi(1 + \phi_E)}{(1 + \phi_E)(\sigma - \mu) + \phi^2 \sigma \left(\frac{1 + \phi_E}{\phi}\right)^{\mu/(\sigma - 1)}} > 1.
\]
(10)

The equality in (10) defines a boundary in the parameter space of the map \( z_d \), denoted \( BT_d1 \):
\[
BT_d1 : \quad \frac{2\sigma \phi(1 + \phi_E)}{(1 + \phi_E)(\sigma - \mu) + \phi^2 \sigma \left(\frac{1 + \phi_E}{\phi}\right)^{\mu/(\sigma - 1)}} = 1.
\]
(11)

When crossing that boundary, the fixed point \( CP_1 \) undergoes a BT bifurcation.

Fig.2a shows a 1D bifurcation diagram of the map \( z_d \) for \( \gamma = 5 \), \( \sigma = 2 \), \( \phi_E = 0.1 \), \( 0 < \phi < 1 \), while Fig.2b and Fig.2c present examples of the map \( z_d \) for various values of \( \phi \). In Fig.2a a fold bifurcation can be recognized leading (for decreasing \( \phi \)) to attracting and repelling fixed points, \( S_a \) and \( S_b \), shown by solid and dashed lines, respectively, which in terms of the map \( Z \) are associated with the symmetric interior fixed points \( S_{aa} \) and \( S_{bb} \). If we continue to decrease \( \phi \) the repelling fixed point \( S_b \) merges quite soon with the fixed point \( CP_1 \) due to the BT bifurcation (using (11) we get that for the considered parameter values this bifurcation occurs for \( \phi \approx 0.135 \)), then the fixed point \( S_a \) undergoes a flip bifurcation. The other two BT bifurcations indicated in Fig.2a occur for the fixed point \( CP_0 \) for increasing \( \phi \) (using (9) we get that these bifurcations occur at \( \phi \approx 0.1891 \) and \( \phi \approx 0.9472 \)).

Fig.3 shows two more 1D bifurcation diagrams of the map \( z_d \), for \( \sigma = 3 \) and \( \sigma = 8 \), where one observes for decreasing \( \phi \) the logistic bifurcation scenario up to the contact of a chaotic attractor with its basin boundary defined by the fixed point \( CP_1 \) and its preimage. This bifurcation (called also final bifurcation) is caused by the homoclinic bifurcation of the fixed point \( S_a \), after which the fixed point \( CP_1 \) becomes an attractor in Milnor’s sense,\(^2\) while the chaotic attractor is transformed into a chaotic repellor.

Fig.4 compares 1D bifurcation diagrams for \( \gamma = 5 \), \( \sigma = 2 \), \( 0 < \phi < 1 \) and \( \phi_E = 0.1 \) in a) and \( \phi_E = 0.001 \) in b). It nicely illustrates that when \( \phi_E \to 0 \), the bifurcation sequence tends to the typical pattern found in standard two-region FE models, where the symmetric fixed point \( x = 0.5 \) undergoes a subcritical pitchfork bifurcation when increasing the trade freeness parameter and a flip bifurcation.

---

\(^2\)Recall that a Milnor attractor is defined as a closed invariant set \( A \subseteq I \) such that its stable set \( \rho(A) \) has a strictly positive measure, and there is no strictly smaller closed subset \( A' \) of \( A \) such that \( \rho(A') \) coincides with \( \rho(A) \) up to a set of measure zero [15]. Differently from a more spread notion of attractor (see e.g., [17], [16]), an attractor in Milnor’s sense does not require existence of a neighborhood any point of which belongs to its basin.

12
Figure 2: a) 1D bifurcation diagram of the map $z_d$ for $\gamma = 5, \sigma = 2, \phi_E = 0.1, 0 < \phi < 1$; the map $z_d$ for $\phi = 0.01, 0.1, 0.15$ is shown in b) and for $\phi = 0.2, 0.4, 0.8$ in c).

Figure 3: 1D bifurcation diagrams of the map $z_d$ for $\gamma = 5, \phi_E = 0.1, 0 < \phi < 1$ and $\sigma = 3$ in a), $\sigma = 8$ in b).
when decreasing the trade freeness parameter. As noted above, in our two-country model with four
regions, \( x = 0.5 \) is a fixed point only in the limiting case \( \phi_E \to 0 \) (implying two separate countries
with two regions with no trade between them). The simulations illustrate that in the general case, the
stable fixed point involves \( x > 0.5 \): because of the access to the international market, as reflected by
the condition \( \phi_E > 0 \), the border region enjoys an advantage compared to the interior region and ends
up with the higher share of entrepreneurs, i.e. with \( x > 0.5 \).

Figure 4: 1D bifurcation diagrams for \( \gamma = 5, \sigma = 2, 0 < \phi < 1 \) and \( \phi_E = 0.1 \) in a), \( \phi_E = 0.001 \) in b).

Next, let us consider the reduction of the map \( Z \) to the invariant line \( L_{x,0} \) (an analogous reduction
obviously holds on the border line \( L_{0,y} \)). It is given by the 1D map defined as follows:

\[
z(0) : x \rightarrow \begin{cases} 
0 & \text{if } z(0)(x) < 0, \\
z(0)(x) & \text{if } 0 \leq z(0)(x) \leq 1, \\
1 & \text{if } z(0)(x) > 1,
\end{cases}
\]

where

\[
z(0)(x) = x + \gamma x(1 - x) \frac{\Omega(x) - 1}{1 + x(\Omega(x) - 1)}, \quad \Omega(x) = \frac{\pi_2}{\pi_1} \left( \frac{\Delta_2}{\Delta_1} \right)^{\mu/(\sigma - 1)},
\]

\[
\Delta_1 = 1 - (1 - \phi)x, \quad \Delta_2 = \phi + (1 - \phi)x, \quad \Delta_3 = \phi + \phi_E x,
\]

\[
\pi_1 = \frac{L \left( \frac{1}{\Delta_1} + \phi \frac{1}{\Delta_3} \right) + \phi x \frac{\pi_2}{\Delta_2}}{\frac{\sigma}{\mu} - \frac{1 - x}{\Delta_1}}, \quad \pi_2 = \frac{B}{A},
\]

\[
A = \frac{\sigma}{\mu} - \frac{x}{\Delta_2} \left( 1 + \frac{\phi^2 (1 - x)}{\Delta_1} \right), \quad B = \frac{L}{2E} \left( \frac{\sigma}{\mu} \frac{\Delta_2 \phi (1 - x)}{\Delta_1 (1 + x)} \phi + \frac{\phi_E}{\Delta_3} + \frac{1}{\Delta_2} \right).
\]
The map $z(0)$, similar to the map $z_d$, has the CP fixed points $CP_0$ and $CP_1$. The fixed point $CP_0$ undergoes the BT bifurcation if $\Omega(0) = 1$ that holds for the same parameter values as those satisfying (9), that is, the boundary in the parameter space, denoted $BT(0)_0$, related to this bifurcation coincides with $BT_{d0}$:

$$BT(0)_0 \equiv BT_{d0}.$$ 

Given that the same conclusion is true for reduction of the map $Z$ to the border $L_{0,y}$, we get

**Proposition 1.** *The inequality (8) is the stability condition for the CP fixed point $CP_{00}$ of the map $Z$.*

From $\Omega(1) > 1$, we get the stability condition of the fixed point $CP_1$ of $z(0)$:

$$\frac{\sigma \phi (2\phi + 3\phi_E)}{(1 + \phi^2)(\sigma - \mu)(\phi + \phi_E) + \mu \phi^2(2\phi + 3\phi_E)} \left( \frac{1}{\phi} \right)^{\mu/(\sigma - 1)} > 1. \quad (12)$$

The BT bifurcation of the fixed point $CP_1$ of the map $z(0)$ occurs if $\Omega(1) = 1$ that holds for

$$BT(0)_1 : \frac{\sigma \phi (2\phi + 3\phi_E)}{(1 + \phi^2)(\sigma - \mu)(\phi + \phi_E) + \mu \phi^2(2\phi + 3\phi_E)} \left( \frac{1}{\phi} \right)^{\mu/(\sigma - 1)} = 1. \quad (13)$$

Fig.5 shows a 1D bifurcation diagram of the map $z(0)$ and its enlargement, as well as examples of this map for various values of $\phi$. Bifurcations observed in the 1D bifurcation diagram are qualitatively similar to those describes above for the map $z_d$.

Now let us consider the reduction of the map $Z$ to the invariant line $L_{x,1}$ (an analogous reduction obviously holds also for $L_{1,y}$). It is given by the 1D map $z(1)$ defined as

$$z(1) : x \rightarrow \begin{cases} 
0 & \text{if } z(1)(x) < 0, \\
z(1)(x) & \text{if } 0 \leq z(1)(x) \leq 1, \\
1 & \text{if } z(1)(x) > 1,
\end{cases}$$

where

$$z(1)(x) = x + \gamma x(1 - x) \frac{\Omega(x) - 1}{1 + x(\Omega(x) - 1)}; \quad \Omega(x) = \frac{\pi_2}{\pi_1} \left( \frac{\Delta_2}{\Delta_1} \right)^{\mu/(\sigma - 1)},$$

$$\Delta_1 = 1 - (1 - \phi)x, \quad \Delta_2 = \phi + (1 - \phi)x + \phi_E, \quad \Delta_3 = 1 + \phi_E x,$$

$$\pi_1 = \frac{L}{2E} \left( \frac{1}{\Delta_1} + \phi \frac{1}{\Delta_2} \right) + \phi x \frac{\pi_2}{\Delta_2}, \quad \pi_2 = \frac{A_f B_h + B_f C_h}{A_h A_f - C_h C_f};$$

$$A_h = \frac{\sigma}{\mu} - \frac{x}{\Delta_2} \left( 1 + \frac{\phi^2(1 - x)}{\mu \Delta_1 - 1 + x} \right), \quad B_h = \frac{L}{2E} \left( \frac{\frac{\sigma}{\mu} \Delta_2 + \phi (1 - x)}{\Delta_2} \phi + \phi_E + 1 \right).$$

15
The map \( z_{(1)} \) has the CP fixed points \( CP_0 \) and \( CP_1 \). From \( \Omega(0) < 1 \) one gets the stability condition of \( CP_0 \):

\[
\frac{\mu \phi_E (\phi_E - \phi) + (\phi + \phi_E)^2 (\sigma + \mu) + (\sigma - \mu)}{(2 \phi + \phi_E) \sigma} (\phi + \phi_E)^{\mu/(\sigma-1)} < 1.
\]

The fixed point \( CP_0 \) undergoes a BT bifurcation if \( \Omega(0) = 1 \) that holds for

\[
BT_{(1)0} : \frac{\mu \phi_E (\phi_E - \phi) + (\phi + \phi_E)^2 (\sigma + \mu) + (\sigma - \mu)}{(2 \phi + \phi_E) \sigma} (\phi + \phi_E)^{\mu/(\sigma-1)} = 1.
\]

The fixed point \( CP_1 \) undergoes a BT bifurcation if \( \Omega(1) = 1 \) that holds for

\[
BT_{(1)1} \equiv BT_{d1}.
\]
Given that the same conclusion is true for reduction of the map \( Z \) on the border \( L_{1,y} \), we get

**Proposition 2.** The inequality (10) is the stability condition for the CP fixed point \( CP_{11} \) of the map \( Z \).

Fig. 6 shows a 1D bifurcation diagram of the map \( z(1) \) for \( \gamma = 5, \sigma = 2, \phi_E = 0.1 \) and \( 0 < \phi < 1 \), as well as examples of this map for various values of \( \phi \). In Fig. 7 two more 1D bifurcation diagrams are shown, for \( \sigma = 3 \) and \( \sigma = 8 \). Bifurcations observed in these 1D bifurcation diagrams are qualitatively similar to those described above for the map \( z_d \).

![Bifurcation Diagram](image)

**Figure 6:** a) 1D bifurcation diagram of map \( z(1) \) for \( \gamma = 5, \sigma = 2, \phi_E = 0.1, 0 < \phi < 1 \); b) the map \( z(1) \) for \( \phi = 0.05, 0.15, 0.4 \).

Using stability condition (14) of the fixed point \( CP_0 \) of the map \( z(1) \) and the stability condition (12) of the fixed point \( CP_1 \) of the map \( z(0) \) we can state

**Proposition 3.** If both the inequalities (14) and (12) are satisfied then the fixed points \( CP_{01} \) and \( CP_{10} \) of the map \( Z \) are attracting.

Figs. 8 and 9 illustrate how we can use the 1D maps \( z_d, z(0) \) and \( z(1) \) to describe the dynamics of the map \( Z \). For the considered parameter values the map \( Z \) has four attracting fixed points, namely, \( CP_{11}, S_{aa}, AS_{c1} \) and \( AS_{1c} \). Their basins of attraction shown in Figs. 8 are separated by the stable sets of the saddle fixed points denoted \( AS_{cf}, AS_{fe}, AS_{1d} \) and \( AS_{d1} \). In this figure as well as in the others,
Figure 7: 1D bifurcation diagram of map $z(1)$ for $\gamma = 5$, $\phi_E = 0.1$, $0 < \phi < 1$ and $\sigma = 3$ in a), $\sigma = 8$ in b).

attracting, repelling and saddle fixed points are marked by black, white and gray circles, respectively. The curves $\Omega_h$ and $\Omega_f$ given in (5) and (6) are also drawn (recall that their mutual intersection points define the interior symmetric and asymmetric fixed points, and the intersection of each of these curves with a border line of $I^2$ defines the related border asymmetric fixed point). In Fig.9 the corresponding maps $z_d$, $z(0)$ and $z(1)$ are shown. From the above analysis, one can deduce that an initial point $(x_0, y_0) \in L_{0,y}$ (see the map $z(0)$) is attracted to the saddle fixed point $CP_{01}$ (similarly, an initial point $(x_0, y_0) \in L_{x,0}$ is attracted to $CP_{10}$), while an initial point $(x_0, y_0) \in L_{x,1}$ (see the map $z(1)$) with $0 < x_0 < d$ is attracted to $AS_{c1}$, and an initial point $(x_0, y_0) \in L_{x,1}$ with $d < x_0 < 1$ is attracted to $CP_{11}$ (analogous conclusions hold for an initial point $(x_0, y_0) \in L_{1,y}$). Finally, an initial point $(x_0, y_0) \in L_d$ (see the map $z_d$) with $0 < x_0 < b$ is attracted to $S_{aa}$, and with $b < x_0 < 1$ is attracted to $CP_{11}$. From an economic point of view, there is a couple of remarkable results: As noted previously, the symmetric fixed point $S_{aa}$ involves $x = y > 0.5$ reflecting the advantage of the border regions due to their access to the international market (indicated by $\phi_E = 0.1 > 0$). More interestingly, two asymmetric, stable fixed points $AS_{c1}$ and $AS_{1c}$ also exist involving partial agglomeration. E.g. in the fixed point $AS_{c1}$, foreign entrepreneurial activity is fully agglomerated in the border region 3, whereas home entrepreneurial activity is dispersed ($c$ indicating the entrepreneurial share in region 2).
Figure 8: Basins of attraction of fixed points $S_{aa}$, $AS_{1c}$, $AS_{c1}$ and $CP_{11}$ of the map $Z$ for $\gamma = 5$, $\sigma = 2$, $\phi_E = 0.1$, $\phi = 0.15$.

Figure 9: The map $z(0)$ in a), $z(1)$ in b) and $z_d$ in c) for $\gamma = 5$, $\sigma = 2$, $\phi_E = 0.1$, $\phi = 0.15$. 
Figure 10: An attracting 2-cycle (yellow circles) and its basin shown in red, and attracting (in Milnor’s sense) fixed points $AS_{c1}$ and $AS_{1c}$ with green and blue basins, respectively, for $\gamma = 5$, $\sigma = 2$, $\phi_E = 0.1$, $\phi = 0.01$.

Figures 10 and 11 illustrate one more example. As can be seen in Fig.10, the map $Z$ has an attracting 2-cycle located on the main diagonal (cf. with the map $z_d$ in Fig.11c) with a basin of attraction shown in red, and two attracting (in Milnor’s sense) fixed points, $AS_{c1} \in L_{x,1}$ and $AS_{1c} \in L_{1,y}$ (cf. with the map $z_{(1)}$ in Fig.11b) with green and blue basins, respectively. Boundaries of these basins are proper segments of $L_{0,y}$, $L_{x,1}$, $L_{x,0}$, $L_{1,y}$ and their preimages. An initial point $(x_0, y_0) \in L_{x,1}$ is attracted to $AS_{c1}$, an initial point $(x_0, y_0) \in L_{1,y}$ is attracted to $AS_{1c}$, while dynamics on $L_{x,0}$ and $L_{0,y}$ are more complicated: as it follows from the dynamics of the map $z_{(0)}$ (see in Fig.11a), an initial point $(x_0, y_0) \in L_{0,y}$ belonging to the segment shown green (related to the flat branch of $z_{(0)}$) or to its preimages is mapped into $CP_{01}$, while other initial points of $L_{0,y}$ (except for the preimages of the fixed point $AS_{0y}$) are attracted to a 2-cycle. Similarly, an initial point $(x_0, y_0) \in L_{x,0}$ belonging to the segment shown blue or its preimages is mapped to $CP_{10}$ while other initial points of $L_{x,0}$ (except for the preimages of the fixed point $AS_{y0}$) are attracted to a 2-cycle belonging to $L_{x,0}$.
3.3 Bifurcation scenarios varying $\phi$

Fig.12 presents a 2D bifurcation diagram and its enlargement in the $(\phi, \gamma)$-parameter plane, together with the bifurcation curves $BT_{d0}$, $BT_{d1}$, $BT_{(0)1}$ and $BT_{(1)0}$ given in (9), (11), (13) and (15), respectively (it is clear that in the $(\phi, \gamma)$-parameter plane these curves represent vertical straight lines as their expressions do not depend on $\gamma$). The horizontal lines with double arrows indicate stability regions of the related CP fixed points. In particular, it can be seen that the blue region bounded by $BT_{d0}$ and $BT_{(1)0}$ is associated with four coexisting attracting CP fixed points. The dark gray region marked $CP_M$ corresponds to CP fixed points which are attractors in Milnor’s sense due to a flat branch in the definition of the map Z. The region shown in yellow is related to attracting border asymmetric fixed points $AS_{x1}$, $AS_{1x}$. The red region marked $S$ is associated with attracting symmetric interior fixed point $S_{xx}$. It can be seen that this fixed point undergoes a flip bifurcation leading to an attracting 2-cycle (its stability region is shown in green). Then, when decreasing $\phi$, a region related to an attracting 4-cycle is recognizable (it is shown in magenta), as well as other periodicity regions. Here, a white region is related to either higher periodicity or chaotic attractors. Given that the 2D bifurcation diagram in Fig.12 is obtained for only one initial point, $(x_0, y_0) = (0.5, 0.51)$, coexistence of attractors other than the CP fixed points cannot be seen in this figure. In order to study such a coexistence we consider below a 1D bifurcation diagram related to the cross-section of the 2D diagram for $\gamma = 5$ indicated by the thick arrow.

First in Fig.13 we present a 1D bifurcation diagram in the $(x, y, \phi)$-space for $\gamma = 5$, $\sigma = 2$, $\phi_E = 0.1$ and $0 < \phi < 1$, where only fixed points are shown, namely, the branches related to the
Figure 12: 2D bifurcation diagram in a) and its enlargement in b) in the $(\phi, \gamma)$-parameter plane for $\sigma = 2$, $\phi_E = 0.1$.

border asymmetric fixed points $AS_{x0}$, $AS_{x1}$, $AS_{0y}$, $AS_{1y}$ are shown in green, blue, magenta and light blue, respectively, while the branches of inner asymmetric and symmetric fixed points $AS_{xy}$ and $S_{xx}$ are shown in red and brown, respectively. Border-transcritical and pitchfork bifurcations that these fixed points undergo, are indicated by black and red circles.

Figure 13: Fixed points of the map $Z$ in the $(x, y, \phi)$-space for $\gamma = 5$, $\sigma = 2$, $\phi_E = 0.1$ and $0 < \phi < 1$. 
Next, in Fig.14 a complete 1D bifurcation diagram is presented in the \((x, y, \phi)\)-space for \(0 < \phi < 0.17\), and its projection on the \((x, \phi)\)-plane is shown in Fig.15. In these figures dashed lines are related to repelling or saddle fixed points, while attracting fixed points are shown by solid lines. Besides border-transcritical and pitchfork bifurcations, fold and flip bifurcations are also indicated. Recall that dynamics associated with the planes \(\{x = 0\} (\{y = 0\})\), \(\{x = 1\} (\{y = 1\})\) and \(\{x = y\}\) are governed by the corresponding 1D maps \(z(0), z(1)\) and \(z_d\) (see Figs.2a, 5a,c and 6a, respectively).

Let us comment now the bifurcation sequence observed for decreasing \(\phi\) at fixed \(\sigma = 2\), \(\phi_E = 0.1\), \(\gamma = 5\) (see the thick arrow in Fig.12), and how basins of attraction of coexisting attractors change due to these bifurcations. We begin with the value \(\phi = 0.4\) at which the map \(Z\) has four coexisting attracting \(CP\) fixed points. Their basins of attraction shown in Fig.16a are separated by the stable sets of the border asymmetric saddle fixed points \(AS_{0b}, AS_{b0}, AS_{c1}\) and \(AS_{1c}\). Decreasing \(\phi\) the basins of \(CP_{00}, CP_{10}\) and \(CP_{01}\) decrease while the basin of \(CP_{11}\) increases (see Fig.16b where \(\phi = 0.25\)). If the parameter point crosses the bifurcation curve \(BT_{d0}\) (at \(\phi \approx 0.1891\)) the fixed point \(CP_{00}\) loses stability, so that between the curves \(BT_{d0}\) and \(BT_{(1)0}\) the map \(Z\) has three coexisting attracting fixed points, \(CP_{11}, CP_{10}\) and \(CP_{01}\). Then crossing \(BT_{(1)0}\) (at \(\phi \approx 0.1655\)) the fixed points \(CP_{10}\) and \(CP_{01}\) become saddles so that between the curves \(BT_{(1)0}\) and \(BT_{d1}\) only \(CP_{11}\) is attracting among the \(CP\) fixed points. However, in this parameter range other fixed points appear.

In fact, at \(\phi \approx 0.1613\) (the parameter point enters the yellow region in Fig.12) a fold bifurcation occurs in the map \(z_{(1)}\) (see the bifurcation marked 1 in Fig.15a) leading to one attracting and one repelling fixed points which in terms of the map \(Z\) are associated with two pairs of border asymmetric fixed points, namely, two attracting fixed points, \(AS_{c1}, AS_{1c}\), and two saddle fixed points, \(AS_{d1}, AS_{1d}\). Thus, after this bifurcation the map \(Z\) has three coexisting attracting fixed points, \(CP_{11}, AS_{c1}\) and \(AS_{1c}\). An example of their basins is shown in Fig.17a where \(\phi = 0.16\).

If we continue to decrease \(\phi\) a fold bifurcation in the map \(z_d\) occurs at \(\phi \approx 0.1565\) (see the bifurcation marked 2 in Fig.15a) leading to a pair of symmetric interior fixed points, a saddle fixed point \(S_{aa}\) and a repelling fixed point \(S_{bb}\). An example of the phase portrait soon after this bifurcation is shown in Fig.17b where \(\phi = 0.155\). Next bifurcation is the subcritical pitchfork bifurcation of \(S_{aa}\) occurring at \(\phi \approx 0.1529\) (the parameter point enters the red region in Fig.12; see also the bifurcation marked 3 in Fig.15a) after which this fixed point becomes attracting and two asymmetric interior saddle fixed points are born, denoted \(AS_{ef}\) and \(AS_{fe}\). An example of the basins of attraction of coexisting fixed
Figure 14: 1D bifurcation diagram of the map $Z$ in the $(x, y, \phi)$-space for $\gamma = 5$, $\sigma = 2$, $\phi_E = 0.1$ and $0 < \phi < 0.17$.

Figure 15: Projection of the 1D bifurcation diagram shown in Fig.14 to the $(x, \phi)$-plane. Here $\gamma = 5$, $\sigma = 2$, $\phi_E = 0.1$ and $0 < \phi < 0.17$ in a), $0 < \phi < 0.03$ in b).
Figure 16: Basins of attraction of $CP$ fixed points $CP_{00}$, $CP_{11}$, $CP_{01}$ and $CP_{10}$ for $\sigma = 2$, $\phi_E = 0.1$, $\gamma = 5$ and $\phi = 0.4$ in a), $\phi = 0.25$ in b).

Figure 17: Basins of attraction of fixed points $CP_{11}$, $AS_{c1}$ and $AS_{1c}$ for $\sigma = 2$, $\phi_E = 0.1$, $\gamma = 5$ and $\phi = 0.16$ in a), $\phi = 0.155$ in b).
points $CP_{11}$, $AS_{c1}$, $AS_{1c}$ and $S_{aa}$ is shown in Fig.8 where $\phi = 0.15$.

Next bifurcation occurs when the parameter point crosses the curve $BT_{d1}$ (at $\phi \approx 0.135$) so that $CP_{11}$ loses stability merging simultaneously with the fixed points $S_{b6}$, $AS_{d1}$ and $AS_{1d}$ due to a BT bifurcation (see the bifurcation marked 4 in Fig.15a). After this bifurcation the map $Z$ has three coexisting fixed points, $S_{aa}$, $AS_{c1}$ and $AS_{1c}$ (see Fig.18a where $\phi = 0.134$) until one more BT bifurcation occurs (see the bifurcation marked 5 in Fig.15a) as $AS_{c1}$ and $AS_{1c}$ lose stability merging with $AS_{ef}$ and $AS_{fe}$, respectively. After this bifurcation the unique attractor is the fixed point $S_{aa}$. At $\phi \approx 0.0256$ it undergoes a flip bifurcation (see the bifurcation marked 8 in Fig.15b). The 2-cycle born due to this bifurcation belongs to the main diagonal $L_d$. See, e.g., Fig.10, where $\phi = 0.01$, which shows an attracting 2-cycle coexisting with attracting (in Milnor’s sense) fixed points $AS_{c1}$ and $AS_{1c}$ (the reason for which these fixed points are attracting again is explained below). At $\phi \approx 0.005$ the 2-cycle undergoes a supercritical pitchfork bifurcation (see the bifurcation marked 10 in Fig.15a) leading to two asymmetric interior 2-cycles which are symmetric to each other with respect to $L_d$ (see Fig.18 where $\phi = 0.004$). Then each of these 2-cycles undergoes a Neimark-Sacker bifurcation leading to closed invariant attracting curves which after destruction lead to chaotic attractors. The pitchfork and Neimark-Sacker bifurcation curves can be seen in Fig.20 which shows 2D bifurcation diagrams in the $(\phi, \phi_{E})$-parameter plane for $\gamma = 5$, $\sigma = 2$.

In the meantime, at $\phi \approx 0.0936$ the map $z(0)$ undergoes a fold bifurcation (see the bifurcation marked 6 in Fig.15a) leading to attracting and repelling fixed points, $AS_g$ and $AS_{b}$, respectively, which in terms of the map $Z$ are associated with two pairs of border asymmetric fixed points, saddles $AS_{0g}$, $AS_{g0}$, and repellors $AS_{0h}$, $AS_{h0}$. Soon after, at $\phi \approx 0.0911$ (the parameter point crosses $BT_{(0)1}$) the fixed points $AS_{0h}$ and $AS_{h0}$ merge with $CP_{01}$ and $CP_{10}$, respectively, due to a BT bifurcation (see the bifurcation marked 7 in Fig.15a), so that saddles $CP_{01}$ and $CP_{10}$ become repelling fixed points. Then the fixed point $AS_g$ of the map $z(0)$ undergoes a cascade of flip bifurcations (two of which are marked 9 and 11 in Fig.15b) following the logistic bifurcation scenario up to the homoclinic bifurcation of $AS_g$ (see Fig.5c and the bifurcation marked 12 in Fig.15b). In the meantime at $\phi \approx 0.0515$ (when the fixed point $AS_g$ is still stable) the flat branch with $x = 1$ appears in the definition of the map $z(0)$ inside the interval $[0, 1]$, so that its fixed point $CP_1$ becomes a Milnor attractor (see, e.g., Fig.11a). In terms of the map $Z$ this leads to the stabilization of fixed points $AS_{c1}$ and $AS_{1c}$ which also become Milnor attractors (see, e.g., Fig.10 or Fig.18b). So, for $\phi \lesssim 0.0515$ all the mentioned above attractors
The previous bifurcation scenario is coherent with the typical sequence found in NEG models: for low values of the (internal) trade freeness, entrepreneurial capital is dispersed over all regions, whereas it is agglomerated for high values of the trade freeness parameter; dispersed and agglomerated attractors may coexist; and the long-run spatial distribution of entrepreneurs sensitively depends upon initial conditions and/or parameters.

At the same time, the patterns that we detect are much richer:

First, we find a much larger variety of fixed point attractors. Dispersion involves a higher share ($x = y > 0.5$) of entrepreneurs in the border regions (due to the access to the foreign market); and agglomeration can now take several forms: $CP_{01}$, $CP_{10}$, $CP_{11}$ and $CP_{00}$; note that $CP_{01}$ and $CP_{10}$ are new asymmetric outcomes, in one country entrepreneurial activity agglomerates in the border region, whereas in the other country it agglomerates in the interior less accessible region. More interestingly and going beyond standard NEG results, for intermediate values of trade freeness, partial agglomeration is also possible: the fixed points $AS_{c1}$ and $AS_{1c}$ involve agglomeration of manufacturing in the border region only in one country, whereas in the other country manufacturing is spread across both regions.

Second, we find that coexistence of equilibria and multistability are much more pervasive than
standard NEG models suggest: For high values of the trade freeness parameter, several core-periphery fixed points are simultaneously stable; there are even parameter ranges in which the four of them are stable. For intermediate values of the trade freeness parameter, fixed points coexist involving full agglomeration, partial agglomeration and symmetric dispersion of entrepreneurial labour. It is interesting to note that the respective basins of attraction as depicted in Fig.17 nicely correspond to the economic intuition: E.g. all initial conditions in the green area involve \( y > x > 0 \); thus, when the initial number of firms is larger in region 3, region 3 benefits from a head start (in the sense of having a larger local market); foreign firms then fully agglomerate in the border region 3 exploiting the market access and the price index effect, whereas home firms in region 2 that start from a less advantageous position partly remain in the interior region in order to avoid competition. The basin of attraction of \( CP_{11} \) is small because of the strong competition effect; only close initial conditions are attracted to it resulting in a symmetric outcome with large local markets. Last, but not least, cyclical coexistence is also possible.

Third, our analysis reveals that some of the fixed points are attracting in Milnor’s sense, which involves complex basins of attraction and therefore a high sensitivity of long-run outcomes on initial conditions.

The described above bifurcation scenario occurs for fixed \( \phi_E = 0.1, \gamma = 5 \) and \( \sigma = 2 \). In Fig.12 it can be seen that for larger values of \( \gamma \) more complicated dynamics occur for larger values of \( \phi \), however, basic bifurcation structures are preserved. To see how this scenario changes if other parameters are varied we show below bifurcation structure in the \((\phi, \sigma)\)- and \((\phi, \phi_E)\)-parameter planes.

Fig.19 presents a 2D bifurcation diagram in the \((\phi, \sigma)\)-parameter plane for \( \phi_E = 0.1, \gamma = 5 \). One can compare the bifurcation sequence described above with the cross-section of this diagram for \( \sigma = 2 \) indicated by the arrow.

Next, in Fig.20 we show a 2D bifurcation diagram in the \((\phi, \phi_E)\)-parameter plane for \( \gamma = 5, \sigma = 2 \). As before, the arrow indicates a cross-section for \( \phi_E = 0.1 \) along which the bifurcation sequence described above is observed. In this parameter plane the pitchfork bifurcation curve of the 2-cycle leading to two coexisting 2-cycles is shown. One can also see the Neimark-Sacker bifurcation curve and \( 1 : 3 \) resonance region which is associated with two coexisting 6-cycles. Basins of two attracting 6-cycles and attracting (in Milnor’s sense) fixed points \( AS_{c1} \) and \( AS_{1c} \) are shown in Fig.21, where \( \phi = 0.003, \phi_E = 0.2 \).
Figure 19: 2D bifurcation diagram in a) and its enlargement in b) in the $(\phi, \sigma)$-parameter plane for $\phi_E = 0.1$, $\gamma = 5$.

Figure 20: 2D bifurcation diagram in a) and its enlargement in b) in the $(\phi, \phi_E)$-parameter plane for $\gamma = 5$, $\sigma = 2$. 


Figure 21: Basins of two attracting 6-cycles (shown in light blue and magenta) and attracting in Milnor’s sense fixed points $AS_{c1}$ and $AS_{1c}$ (shown in green and dark blue, resp.) for $\gamma = 5$, $\sigma = 2$, $\phi = 0.003$, $\phi_E = 0.2$.

4 Conclusion

In this paper, we analysed a New Economic Geography model with two countries, each of which consisting of two regions. We introduced a specific geographical structure by considering regions arranged along a line. This geographical set-up has specific consequences for the interpretation of agglomerating and spreading forces: In the interior regions (resp. 1 and 4), competition comes from local firms and - to a lesser extent because of the transport costs - from firms located in the neighbouring border region (resp. 2 and 3). In the border regions (2 and 3), competition is stronger, because it stems not only from firms in the region itself and in the neighbouring interior region, but also from firms in the border region of the other country, with trade costs suitably accounted for. Thus, competition is more intense in border regions, which puts firms in the border regions on a disadvantage. However, the border regions also provide an access to a larger market because firms located there do not only sell to both regions of the own country (as do the firms located in the interior region), but they are also able to sell to the foreign border region. In addition, the price index effect also favours border
regions, since consumers residing there have access to the varieties offered in the own country (as do consumers residing in the interior region), but also to foreign variants. Therefore, all three effects - competition, market size, and price index - are stronger in border regions than in interior regions; in particular, their strength depends on the size of the manufacturing sector in the other country’s border region. The larger it is, the stronger the three effects. These effects also depend on the external trade cost. The lower the external trade cost, the stronger all three effects.

In our 2-country model with four regions, we detected a sequence that is typical in tomahawk diagrams of standard NEG models: dispersion for low values of the trade freeness and agglomeration for high values of the trade freeness. However, the patterns we found are much richer: dispersion involves a higher share (> 0.5) of industrial activity in the border regions (because of the access to the foreign market); and agglomeration can take several forms including asymmetric outcomes. Most interestingly, for intermediate values of the trade freeness, partial agglomeration is also possible: some fixed points involve the agglomeration of industrial activity in the border region in one country only whereas industrial activity is spread across both regions in the other country. This is explained by the fact that the interior region offers shelter from foreign competition. In addition, we find many instances of coexistence of equilibria and sensitive dependence on initial conditions.

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References


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